

Cubical Covers of Sets in \mathbb{R}^n

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Abstract

Wild sets in \mathbb{R}^n can be tamed through the use of various representations though sometimes this taming removes features considered important. Finding the wildest sets for which it is still true that the representations faithfully inform us about the original set is the focus of this rather playful, expository paper that we hope will stimulate interest in cubical coverings as well as the other two ideas we explore briefly: Jones' β numbers and varifolds from geometric measure theory.

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1 Introduction

In this paper we explain and illuminate a few ideas for (1) representing sets and (2) learning from those representations. Though some of the ideas and results we explain may be written down elsewhere (we are not aware of those references!), our main purpose is not to claim priority, but rather to stimulate thought and exploration. Our primary intended audience is students of mathematics even though other, more mature mathematicians may find a few of the ideas interesting. We believe that cubical covers can be used at an earlier point in the student career and that both the β numbers idea introduced by Peter Jones and the idea of varifolds pioneered by Almgren and Allard are still very much underutilized by all (young and old!). To that end, we have written this exploration, hoping that the questions and ideas presented here, some rather elementary, will stimulate others to explore the ideas for themselves.

We begin by briefly introducing all three approaches – cubical covers, Jones' β , and varifolds, after which we look more closely at questions involving cubical covers. Then both of the other approaches are explained in a little bit of detail, mostly as an invitation to more exploration, after which we close with problems for the reader and some unexplored questions.

Note on measures: If the reader is not well acquainted with k -dimensional Hausdorff measure in \mathbb{R}^n , which we denote by \mathcal{H}^k , little harm will come, for the purposes of this paper, in realizing that this is just a sensible extension of the notion of the usual measure in \mathbb{R}^k to the case in which we insert pieces of \mathbb{R}^k into \mathbb{R}^n and let those pieces deform. For example, one certainly has no problem understanding what the length of a curve in \mathbb{R}^3 is. That would be what \mathcal{H}^1 computes. And of course, \mathcal{L}^n in \mathbb{R}^n is the usual n -dimensional measure in \mathbb{R}^n . For a more careful definition of Hausdorff measure \mathcal{H}^k (and Lebesgue measure \mathcal{L}^n), see the references and comments in the last section of the paper.

2 Representing Sets & their Boundaries in \mathbb{R}^n

2.1 Cubical Refinements: Dyadic Cubes

In order to characterize various sets in \mathbb{R}^n , we wish to explore the use of cubical covers, especially those whose cubes have side lengths which are positive integer powers of $\frac{1}{2}$, **dyadic cubes**, or more precisely, (closed) dyadic n -cubes with sides parallel to the axes. We start with looking at a unit cube in \mathbb{R}^2 lying in the first quadrant with a vertex at the origin. We then form a sequence of refinements by dividing each side length in half successively, and thus quadrupling the number of cubes each time, as shown in Figure 1. This allows us to let the exponent d in the side length $l(C) = \frac{1}{2^d}$ go to infinity when performing limit operations.

Definition 2.1. We shall say that the n -cube C (with side length denoted as $l(C)$) is dyadic [23] if

$$C = \prod_{j=1}^n [m_j 2^{-d}, (m_j + 1) 2^{-d}], \quad m_j \in \mathbb{Z}, d \in \mathbb{N}.$$

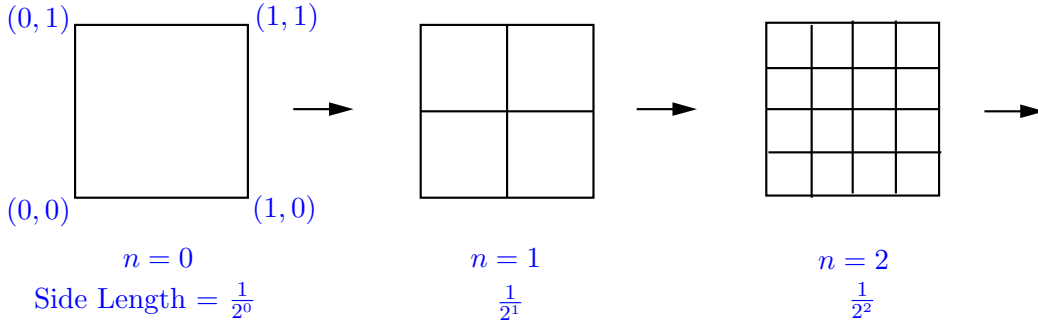


Figure 1: Dyadic Cubes

We will denote the union of the cubes with edge length $\frac{1}{2^d}$ that intersect the set E by \mathcal{C}_d^E .

Questions we will explore include

1. “If I know $\mathcal{L}^n(\mathcal{C}_d^E)$, what can I say about $\mathcal{L}^n(E)$?” and similarly,
2. “If I know $\mathcal{H}^{n-1}(\partial\mathcal{C}_d^E)$, what can I say about $\mathcal{H}^{n-1}(\partial E)$?”

2.2 Jones’ β Numbers

Another approach to representing sets in \mathbb{R}^n developed by Jones [16] and generalized by Okikiolu [23], Lerman [18], and Schul [25], involves the question of under what

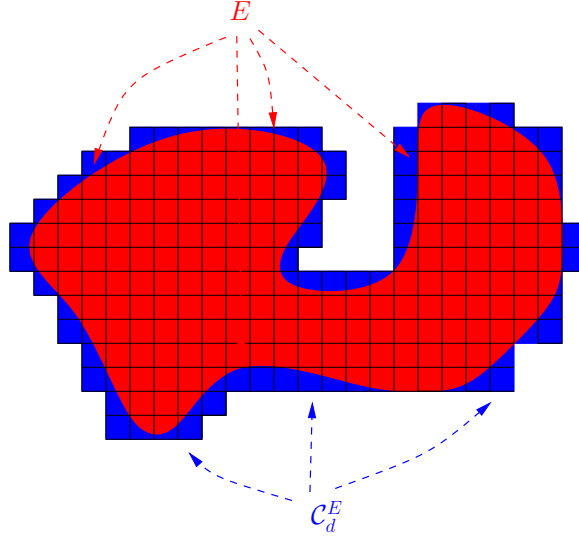


Figure 2: Cubical cover of a set E

conditions can a bounded set E be contained within a rectifiable curve Γ , which Jones likened to the Traveling Salesman Problem taken over an infinite set. If we recall that a connected set $\Gamma \subset \mathbb{R}^2$ is a rectifiable curve if and only if $l(\Gamma) = \mathcal{H}^1(\Gamma) < \infty$, Jones' main result (stated below) can be derived from the following theorem:

Theorem 2.1. *Let E be a bounded set and Γ be a connected set both in \mathbb{R}^2 . Define $\beta_\Gamma(C) \equiv \frac{W_C}{l(C)}$, where W_C is the width of the thinnest cylinder in the n -cube C containing Γ . Then, summing over all possible C ,*

$$\beta^2(\Gamma) \equiv \sum_C (\beta_E(3C))^2 l(C) < \alpha l(\Gamma) < \infty, \text{ where } \alpha \in \mathbb{R}.$$

Conversely, if $\beta^2(E) < \infty$ there is a connected set $\Gamma, E \subset \Gamma$, such that

$$l(\Gamma) \leq (1 + \delta) \text{diam}(E) + \alpha_\delta \beta^2(E).$$

See [16] for a proof. It should be noted that Jones actually worked in \mathbb{C} , but the work of Okikiolu allows us to apply the result to $\mathbb{R}^n \forall n \in \mathbb{N}$.

The mysterious quantity $\beta^2(\Gamma)$ is derived from what are now known as Jones' β numbers, which are defined on each dyadic cube (for fixed n). Loosely speaking, these β numbers depend on the width of the thinnest cylinder in each C containing Γ . Thus Theorem 2.1 and its generalized counterpart provide one approach of defining a useful quantity for a connected set in \mathbb{R}^n similar to those discussed in the next section. We return to this topic in Section 8 to define β numbers more precisely and state Jones' main result referred to above.

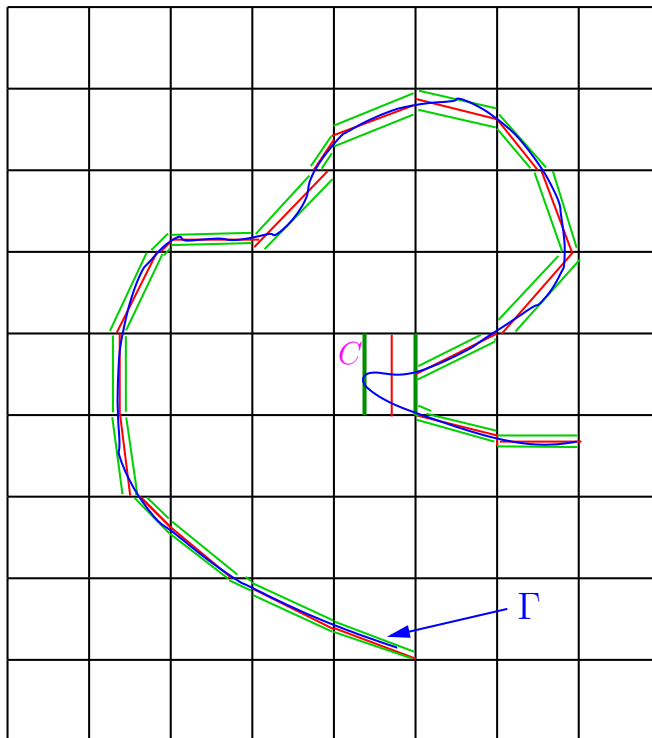


Figure 3: Jones' β Numbers. The green lines indicate the thinnest cylinder containing Γ in cube C . We see from this relatively large width that Γ is not very “flat” in this cube.

2.3 Working Upstairs: Varifolds

We briefly present a third way of representing sets in \mathbb{R}^n that draws out certain properties more readily than viewing sets strictly in \mathbb{R}^n , for example if we want to “build in” a way to determine how “close” or how similar two sets are that takes tangents (or more generally, tangent cones) into account. We outline this approach here and provide further details in Section 8.2.

This technique involves the use of varifolds, which are radon measures defined on $\mathbb{R}^n \times G(n, m)$, the space of the Grassmann bundle over \mathbb{R}^n . The *Grassmannian*, as it's called, consists of all m -dimensional linear subspaces of an n -dimensional vector space, in this case \mathbb{R}^n . So for $G(2, 1)$ it is the space of all lines through the origin in \mathbb{R}^2 , and for $G(3, 2)$, it is the space of all planes through the origin in \mathbb{R}^3 .

Suppose μ is a varifold, $S = \text{spt}(\mu)$, and $\pi : (x, g) \in \mathbb{R}^n \times G(n, m) \rightarrow x$. We will call $E = \pi(S)$ the “downstairs” portion of S . Conversely, suppose E is an

m -rectifiable set (see Definition 5.2) in \mathbb{R}^n . Define $S \subset \mathbb{R}^n \times G(n, m)$ by $S \equiv \{(x, T_x E) \mid x \in E\}$. The measure $\mu_S \equiv \mathcal{H}^m \llcorner S$ is a varifold equal to E “downstairs.”

Alternatively $\mu_s(A) \equiv \mathcal{H}^m(\pi(A))$ gives another varifold based on S . This time, yielding a measure that projects to $\mathcal{H}^m \llcorner E$.

We can parameterize the Grassmannian $G(2, 1)$ by taking the exact upper unit semicircle in \mathbb{R}^2 (including the point $(1, 0)$, but not including $(1, \pi)$, where both points are given in polar coordinates) and straightening it out into a vertical axis (as in Figure 4). The bundle $\mathbb{R}^2 \times G(2, 1)$ is then represented by $\mathbb{R}^2 \times [0, \pi)$.

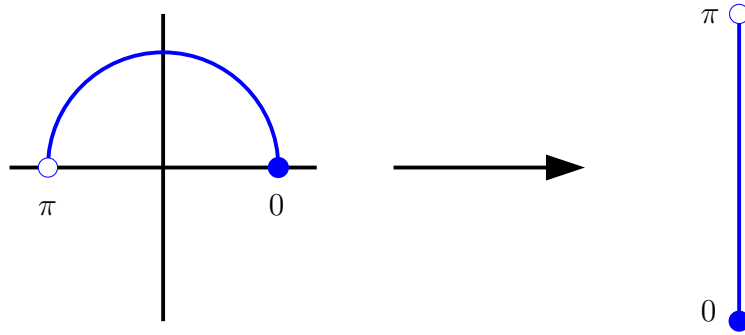


Figure 4: The vertical axis for the “upstairs.”

In this case, the angle of the tangent to E at any given x is then plotted on this vertical axis, with 0 slope having a height of 0 in $E \times G(n, m)$; a vertical slope corresponding to a height of $\frac{\pi}{2}$; and negative slopes approaching 0 having a height approaching π .

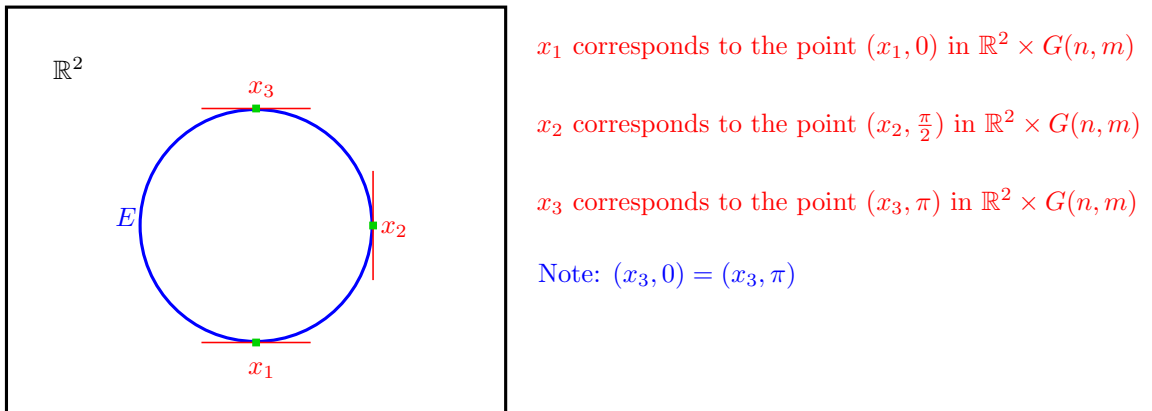


Figure 5: Parameterizing E

This takes a circular curve and maps it to two half-spirals upstairs, as shown in the first image of Figure 6.

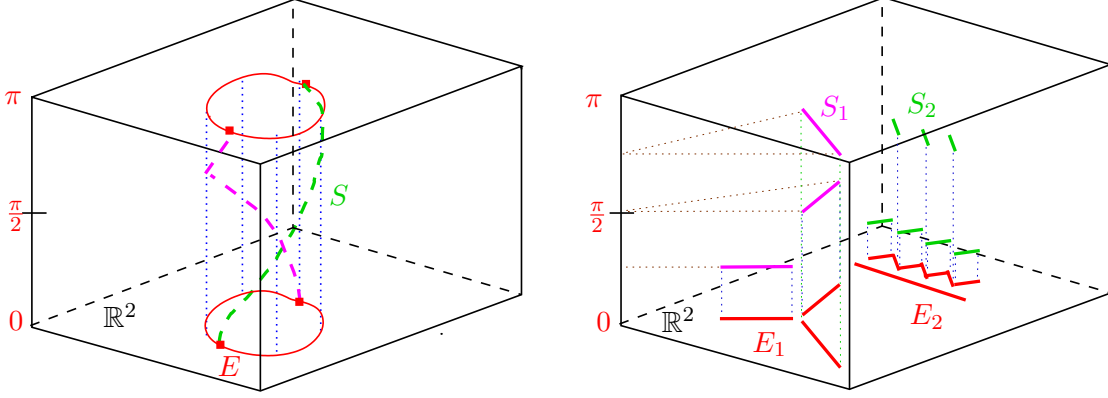


Figure 6: Working Upstairs

We now see that tangents are “built in” to this representation of sets in \mathbb{R}^n . Thus if we think of two perpendicular line segments almost touching each other, i.e. appearing “close” in \mathbb{R}^2 , with one having 0 slope and the other undefined slope, we see that the “upstairs” view keeps the segment with 0 slope at a height of 0, but the other will actually be raised to a height of $\frac{\pi}{2}$. So, these two segments are not very “close” at all in $\mathbb{R}^2 \times G(n, m)$! Or, a straight line segment and a very fine “sawtooth” segment may look practically indistinguishable, but will appear drastically different upstairs.

3 A Simple Question

Suppose $\mathcal{C}_d^E = \{C \mid C \cap E \neq \emptyset, \text{edge length of } C = \frac{1}{2^d}\}$

Question: When can we say

$$\mathcal{L}^n(\mathcal{C}_d^E) \leq M(n)\mathcal{L}^n(E) \quad (1)$$

for some constant $M(n)$ independent of E , d large enough? Or

$$\mathcal{L}^n(\mathcal{C}_d^E) \leq (1 + \delta)\mathcal{L}^n(E) \quad (2)$$

for any $\delta > 0$ as long as d is large enough?

Example 3.1. If $E = \mathbb{Q}^n \cap [0, 1]^n$, then $\mathcal{L}^n(E) = 0$ but $\mathcal{L}^n(\mathcal{C}_d^E) = 1 \ \forall d \geq 0$.

Example 3.2. Let E be as above. Enumerate E as $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \dots$. Now let $D_i = B(\mathbf{q}_i, \frac{\epsilon}{2^i})$ and $E_\epsilon \equiv \{\cup D_i\} \cap [0, 1]^n$ with ϵ chosen small enough so that $\mathcal{L}^n(E_\epsilon) \leq \frac{1}{100}$. Then $\mathcal{L}^n(E_\epsilon) \leq \frac{1}{100}$, but $\mathcal{L}^n(\mathcal{C}_d^{E_\epsilon}) = 1 \ \forall d > 0$.

4 A Union of Balls

For a given set $F \subseteq \mathbb{R}^n$, suppose $E = \cup_{x \in F} B(x, r)$, a union of closed balls of radius r centered at each point x in F . Then we know that E is **regular** (locally Ahlfors n -regular or *locally n -regular*), and thus

$$M^{-1}r^n \leq \mathcal{L}^n(B(x, r) \cap E) \leq Mr^n$$

for some $M \geq 1, \forall x \in E$, and $\forall r \in (0, < r_0)$ for some r_0 . This is all we need to establish a sufficient condition for Equation (1) above. (Note that the upper bound is trivial, which is not the case for k -regular sets in \mathbb{R}^n .)

1. Let $\mathcal{C} = \mathcal{C}_d^E$ for some d such that $\frac{1}{2^d} \ll r$, and let $\hat{\mathcal{C}} = \{3C \mid C \in \mathcal{C}\}$, where $3C$ is an n -cube concentric with C with sides parallel to the axes and $l(3C) = 3l(C)$.

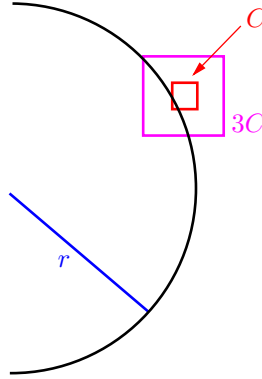


Figure 7: Concentric Cubes

2. This implies that for $3C \in \hat{\mathcal{C}}$

$$\frac{\mathcal{L}^n(3C \cap E)}{\mathcal{L}^n(3C)} > \theta > 0, \quad \text{with } \theta \in \mathbb{R}. \quad (3)$$

3. We then make the following observations:

- (a) $\mathcal{C} \subset \hat{\mathcal{C}}$
- (b) There are 3^n different shifts of any $3C$ n -cube that will still contain C , with each shift tiling \mathbb{R}^n and each cube in $\hat{\mathcal{C}}$ containing at least $\theta \mathcal{L}^n(3C)$ of E .
- (c) We can split $\hat{\mathcal{C}}$ into 3^n disjoint sets of cubes (that is, their interiors are disjoint), each of which contains at most $N = \frac{\mathcal{L}^n(E)}{\theta \mathcal{L}^n(3C)}$ cubes. Thus if we denote the total number of cubes in \mathcal{C} by $N_{\mathcal{C}_d^E}$, then

$$\begin{aligned}
\mathcal{L}^n(\mathcal{C}) &= N_{\mathcal{C}_d^E} \mathcal{L}^n(C) \\
&\leq 3^n N \mathcal{L}^n(C) \\
&= \frac{3^n \mathcal{L}^n(E) \mathcal{L}^n(C)}{\theta \mathcal{L}^n(3C)} \\
&= \frac{\mathcal{L}^n(E)}{\theta}.
\end{aligned} \tag{4}$$

5 Minkowski Content

Definition 5.1. Minkowski Content: Let $W \subset \mathbb{R}^n$, and let $W_r \equiv \{x \mid d(x, W) < r\}$. The $(n-1)$ -dimensional Minkowski Content is defined as $\mathcal{M}^{n-1}(W) \equiv \lim_{r \rightarrow 0} \frac{\mathcal{L}^n(W_r)}{2r}$, when the limit exists.

Definition 5.2. Rectifiable Set: A set $W \subset \mathbb{R}^n$ is called (\mathcal{H}^m, m) rectifiable if $\mathcal{H}^m(W) < \infty$ and \mathcal{H}^m almost all of W is contained in the union of the images of countably many Lipschitz functions from \mathbb{R}^m to \mathbb{R}^n .

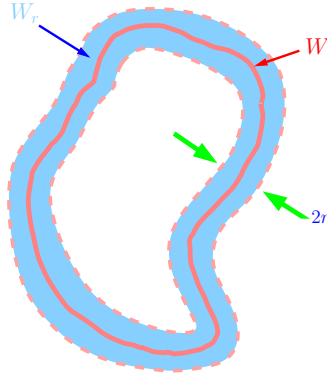


Figure 8: Minkowski Content

Theorem 5.1. $\mathcal{M}^{n-1}(W) = \mathcal{H}^{n-1}(W)$ when W is a closed rectifiable set.

See [15] for a proof.

Now, let W be $(n-1)$ -rectifiable, set $r_d \equiv \sqrt{n} \left(\frac{1}{2^d} \right)$, and choose r_δ small enough so that

$$\mathcal{L}^n(W_{r_d}) \leq \mathcal{M}^{n-1}(W_{r_d})2r_d + \delta \quad \forall r_d \leq r_\delta.$$

We choose r_d in this way due to the following. Suppose we have an n -cube intersecting E in exactly 1 point. Then the farthest distance away from E any other point of this cube can be is the length of the diagonal of the cube, which is precisely $\sqrt{n} \left(\frac{1}{2^d} \right)$.

Letting $W \equiv \partial E$, we have

$$\begin{aligned} \mathcal{L}^n(\mathcal{C}_d^E) - \mathcal{L}^n(E) &\leq \mathcal{L}^n(W_{r_d}) \\ &\leq \mathcal{M}^{n-1}(\partial E)2r_d + \delta \\ &\leq \mathcal{M}^{n-1}(\partial E)2r_\delta + \delta \end{aligned}$$

so that

$$\mathcal{L}^n(\mathcal{C}_d^E) \leq (1 + \hat{\delta})\mathcal{L}^n(E), \quad \text{where } \hat{\delta} = \frac{\mathcal{M}^{n-1}(\partial E)2r_\delta + \delta}{\mathcal{L}^n(E)}. \quad (5)$$

Since we control r_δ and δ , we can send $\hat{\delta}$ to 0, and we have a sufficient condition to establish Equation (2) above.

Problem 5.1. Show that the boundary of a union of closed balls, whose centers are a closed, bounded set, is rectifiable and closed and has finite \mathcal{H}^{n-1} measure.

Corollary 5.1. We can cover unions of open balls of radius r whose centers are bounded with a cover satisfying Equation (2).

6 Smooth Boundary, Positive Reach

If ∂E is *smooth* (at least $C^{1,1}$), ∂E has positive *reach* and therefore, as we shall see, we get an even cleaner bound, depending only on local properties of the boundary.

Definition 6.1. The **reach** of E , $\text{reach}(E)$, is defined as $\text{reach}(E) = \sup\{r \in \mathbb{R} \mid \forall x \in \mathbb{R}^n \setminus E \text{ with } \text{dist}(x, E) < r, \text{ there exists a unique closest point } y \in E \text{ such that } \text{dist}(x, y) = \text{dist}(x, E)\}$.

Equivalently, the $\text{reach}(E)$ is the infimum of the distances one must go from the set to find two normals based at different points on E intersecting. Alternatively, the $\text{reach}(E)$ of a set in \mathbb{R}^n is the supremum of radii r of the balls such that each ball of that radius touching the inside or outside of ∂E only touches the set at one point.

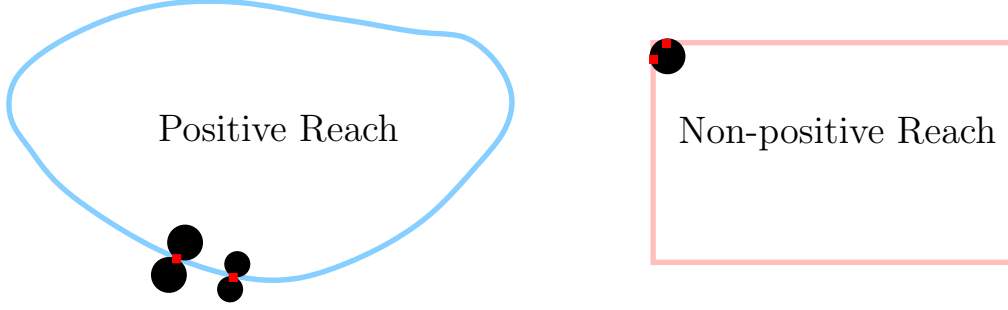


Figure 9: Positive and Non-positive Reach

If the set ∂E is $C^{1,1}$, then it has positive reach. That is, if for all $x \in \partial E$, there is a neighborhood of x , $U_x \subset \mathbb{R}^n$, such that after a suitable change of coordinates, there is a $C^{1,1}$ function $f : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ such that $\partial E \cap U_x$ is the graph of f . (Recall that a function is $C^{1,1}$ if its derivative is Lipschitz continuous.) This implies, among other things, that the (symmetric) second fundamental form of ∂E exists \mathcal{H}^{n-1} almost everywhere on ∂E . The fact that ∂E is $C^{1,1}$ implies that at \mathcal{H}^{n-1} almost every point of ∂E , the $n - 1$ principal curvatures κ_i for $1 \leq i \leq n - 1$ of our set exist and, for all i , $|\kappa_i| \leq \frac{1}{\text{reach}(\partial E)}$.

We will use this fact to determine a bound for the $(n - 1)$ -dimensional change in area as the boundary of our set is expanded outwards or contracted inwards by ϵ . (See Figure 10, Diagram 1). Let us first look at this in \mathbb{R}^2 by examining the following ratios of lengths of expanded or contracted arcs for sectors of a ball in \mathbb{R}^2 as shown in Diagram 2 in Figure 10 below.

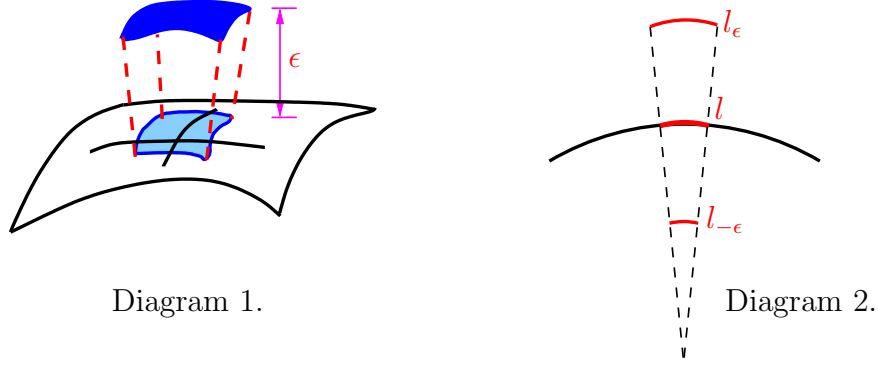


Figure 10: Moving Out and Sweeping In

$$\frac{\mathcal{H}^1(l_\epsilon)}{\mathcal{H}^1(l)} = \frac{(r + \epsilon)\theta}{r\theta} = 1 + \frac{\epsilon}{r} = 1 + \epsilon\kappa$$

$$\frac{\mathcal{H}^1(l_{-\epsilon})}{\mathcal{H}^1(l)} = \frac{(r - \epsilon)\theta}{r\theta} = 1 - \frac{\epsilon}{r} = 1 - \epsilon\kappa,$$

where κ is the principal curvature of the circle (the boundary of the 2-ball), which we can think of as defining the reach of a set $E \subset \mathbb{R}^2$ with $C^{1,1}$ -smooth boundary.

The Jacobian for the normal map pushing in or out by ϵ , which by the area formula is the factor by which the area changes, is given by $\prod_{i=1}^{n-1} (1 + \epsilon\kappa_i)$. (See Figure 10, Diagram 1.) If we define $\hat{\kappa} \equiv \max\{|\kappa_1|, |\kappa_2|, \dots, |\kappa_{n-1}|\}$, then the Jacobian of the normal map that pushes the boundary in or out by ϵ is given by $\prod_{i=1}^{n-1} (1 - \epsilon\hat{\kappa})$ or $\prod_{i=1}^{n-1} (1 + \epsilon\hat{\kappa})$, respectively, and we have the following bounds.

Max Fractional Increase of \mathcal{H}^{n-1} boundary “area” Moving Out:

$$\prod_{i=1}^{n-1} (1 + \epsilon\kappa_i) \leq (1 + \epsilon\hat{\kappa})^{n-1}$$

Max Fractional Decrease of \mathcal{H}^{n-1} boundary “area” Sweeping In:

$$\prod_{i=1}^{n-1} (1 - \epsilon\kappa_i) \geq (1 - \epsilon\hat{\kappa})^{n-1}$$

Remark 6.1. Notice that $\hat{\kappa} = \frac{1}{\text{reach}(\partial E)}$

For a ball, we readily find the value of the ratio

$$\frac{\mathcal{H}^n(B(0, r + \epsilon))}{\mathcal{H}^n(B(0, r))} = \left(\frac{r + \epsilon}{r} \right)^n \quad (6)$$

$$= (1 + \epsilon\kappa)^n \text{ (setting } \delta = \epsilon\kappa) \quad (7)$$

$$= (1 + \delta)^n \quad (8)$$

where $\kappa = \frac{1}{r}$ is the curvature of the ball along any geodesic.

Now we calculate the bound we are interested in for E , assuming ∂E is $C^{1,1}$. Define $E_\epsilon \subset \mathbb{R}^n \equiv \{x \mid d(x, E) < \epsilon\}$. We first compute a bound for

$$\begin{aligned} \frac{\mathcal{H}^n(E_\epsilon)}{\mathcal{H}^n(E)} &= \frac{\mathcal{H}^n(E) + \mathcal{H}^n(E_\epsilon \setminus E)}{\mathcal{H}^n(E)} \\ &= 1 + \frac{\mathcal{H}^n(E_\epsilon \setminus E)}{\mathcal{H}^n(E)}. \end{aligned} \quad (9)$$

Computing bounds for the numerator and denominator separately in the second term in (9), we find

$$\begin{aligned} \mathcal{H}^n(E_\epsilon \setminus E) &= \int_0^\epsilon \int_{\partial E} \prod_{i=1}^{n-1} (1 + r\kappa_i) d\mathcal{H}^{n-1} dr \\ &\leq \int_0^\epsilon \int_{\partial E} (1 + r\hat{\kappa})^{n-1} d\mathcal{H}^{n-1} dr \\ &= \mathcal{H}^{n-1}(\partial E) \left. \frac{(1 + r\hat{\kappa})^n}{n\hat{\kappa}} \right|_0^\epsilon \\ &= \mathcal{H}^{n-1}(\partial E) \left(\frac{(1 + \epsilon\hat{\kappa})^n}{n\hat{\kappa}} - \frac{1}{n\hat{\kappa}} \right) \end{aligned} \quad (10)$$

$$\begin{aligned} \mathcal{H}^n(E) &\geq \int_0^{r_0} \int_{\partial E} \prod_{i=1}^{n-1} (1 - r\kappa_i) d\mathcal{H}^{n-1} dr \\ &\geq \int_0^{r_0} \int_{\partial E} (1 - r\hat{\kappa})^{n-1} d\mathcal{H}^{n-1} dr \\ &= \mathcal{H}^{n-1}(\partial E) \left. \frac{-(1 - r\hat{\kappa})^n}{n\hat{\kappa}} \right|_0^{r_0} \\ &= \frac{\mathcal{H}^{n-1}(\partial E)}{n\hat{\kappa}} \quad \text{when } r_0 = \frac{1}{\hat{\kappa}}. \end{aligned} \quad (11)$$

From 9, 10, and 11, we have

$$\begin{aligned}
\frac{\mathcal{H}^n(E_\epsilon)}{\mathcal{H}^n(E)} &\leq 1 + \frac{\mathcal{H}^{n-1}(\partial E) \left(\frac{(1+\epsilon\hat{\kappa})^n}{n\hat{\kappa}} - \frac{1}{n\hat{\kappa}} \right)}{\frac{\mathcal{H}^{n-1}(\partial E)}{n\hat{\kappa}}} \\
&= (1 + \epsilon\hat{\kappa})^n \quad (\text{setting } \delta = \epsilon\hat{\kappa}) \\
&= (1 + \delta)^n.
\end{aligned} \tag{12}$$

Thus we find a nice, simple bound (the same bound as in the very simple case of balls!) on $\mathcal{H}^n(E_\epsilon)$ when ∂E is smooth enough to have positive reach.

7 A Boundary Conjecture

What can we say about boundaries? Can we bound

$$\frac{\mathcal{H}^{n-1}(\partial \mathcal{C}_d^E)}{\mathcal{H}^{n-1}(\partial E)}?$$

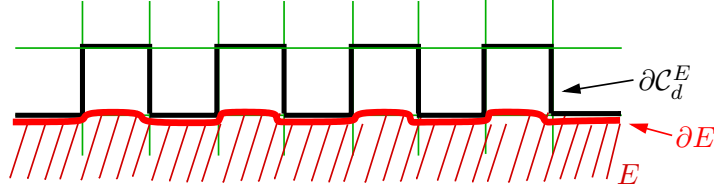


Figure 11: Cubes on the Boundary

Conjecture 7.1. *If $E \subset \mathbb{R}^n$ has positive reach, then*

$$\limsup_{d \rightarrow \infty} \frac{\mathcal{H}^{n-1}(\partial \mathcal{C}_d^E)}{\mathcal{H}^{n-1}(\partial E)} \leq n.$$

Brief Sketch of Proof for $n = 2$.

1. Since ∂E is $C^{1,1}$ then we can zoom in far enough at any point $x \in \partial E$ so that it looks flat. Rotate so that the tangent at x is horizontal.
2. Let C be a cube in the cover \mathcal{C}_d^E that intersect the boundary near x and has faces in the boundary $\partial \mathcal{C}_d^E$. Define $F = \partial C \cap \partial \mathcal{C}_d^E$.
3. (Case 1) Assume that the tangent at x , $T_x \partial E$, is not parallel to either edge direction of the cubical cover. (See Figure 12.)

- (a) Let Π be the projection onto the horizontal axis and notice that $\frac{\mathcal{H}^1(F)}{\Pi(F)} \leq 2 + \epsilon$ for any epsilon.
 - (b) This is stable to perturbations which is important since the actual piece of the boundary ∂E we are dealing with is not a straight line.
4. (Case 2) Suppose that the tangent at x , $T_x \partial E$, is parallel to one of the edge directions of the cubical cover.
- (a) Zooming in far enough, we see that the cubical boundary can only oscillate up and down so that the maximum ratio for any horizontal tangent is (locally) 2.
 - (b) But we can create a sequence of examples that attain ratios as close to 2 as we like by finding a careful sequence of perturbations that attains a ratio locally of $2 - \epsilon$ for any ϵ . (See Figure 11.)
 - (c) That is, we can create perturbations that, on an unbounded set of d 's, $\{d_i\}_{i=1}^\infty$, yield a ratio $\frac{\mathcal{H}^1 \mathcal{C}_{d_i}^E \cap U_x}{\partial E} > 2 - \epsilon$, and we can send $\epsilon \rightarrow 0$.
5. Use the compactness of ∂E to put this all together into a complete proof.

□

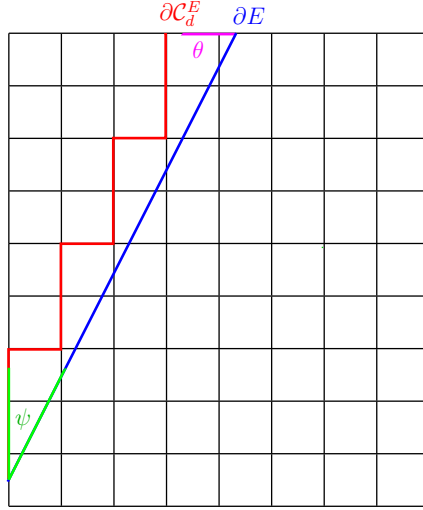


Figure 12: The case where ∂E intersects $\partial \mathcal{C}_d^E$ with $\theta \neq 0, \pi$.

Problem 7.1. Suppose we exclude C 's that contain less than some fraction θ of E from the cover to get the reduced cover $\hat{\mathcal{C}}_d^E$. In this case, what is the optimal bound $B(\theta)$ for the this ratio of boundary measures

$$\limsup_{d \rightarrow \infty} \frac{\mathcal{H}^{n-1}(\partial \hat{\mathcal{C}}_d^E)}{\mathcal{H}^{n-1}(\partial E)} \leq B(\theta)?$$

8 Other Representations

8.1 The Jones' β Approach

We now formally define Jones' β numbers, stated in Theorem 2.1. Let W_C denote the width of the thinnest cylinder containing the curve Γ (or set E) in the dyadic n -cube C , and define the β number of Γ in C to be

$$\beta_\Gamma(C) \equiv \frac{W_C}{l(C)}.$$

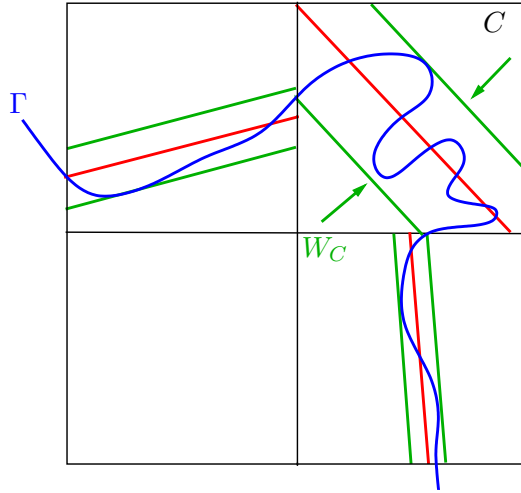


Figure 13: Jones' β Numbers and W_C . The red lines are “best-fit” lines in the sense that for a given dyadic n -cube C , each green line is an equal distance away from each red line so that each pair of green lines defines the thinnest cylinder in C .

Jones' main result [16], derived from Theorem 2.1 and generalized to \mathbb{R}^n , is that a bounded set $E \subset \mathbb{R}^n$ is contained in a rectifiable curve Γ if and only if

$$\beta^2(E) \equiv \sum_C (\beta_E(3C))^2 l(C) < \infty,$$

where the sum is taken over all dyadic cubes. Note that each β number of E is calculated over the dyadic cube $3C$, as defined in Section 4. Intuitively, we see that in order for E to lie within a rectifiable curve Γ , E must look “flat” as we zoom in on points of E since Γ has tangents at \mathcal{H}^1 -almost every point of Γ . Thus W_C above is equal to $\beta_E(C)l(C)$, and we find that $\beta_E(C)$ serves as a scale-invariant notion of “flatness.”

8.2 A Varifold Approach

Let us more formally define what was described in Section 2.3.

$$\begin{aligned} G(n, m) &= G(\mathbb{R}^n, m) \\ &= m\text{-dimensional Grassmannian} \\ &= \text{set of all } m\text{-dimensional planes through the origin.} \end{aligned}$$

Let $\mathbb{R}^n \times G(n, m)$ be the Grassmann bundle, which can be thought of as a space where $G(n, m)$ is attached to each point in \mathbb{R}^n .

Or, we can work in $\mathbb{R}^n \times G_{+/-}(n, m)$, where

$$\begin{aligned} G_{+/-}(n, m) &= G(n, m), \text{ oriented} \\ &= \text{set of all unit } m\text{-vectors in } \mathbb{R}^n. \end{aligned}$$

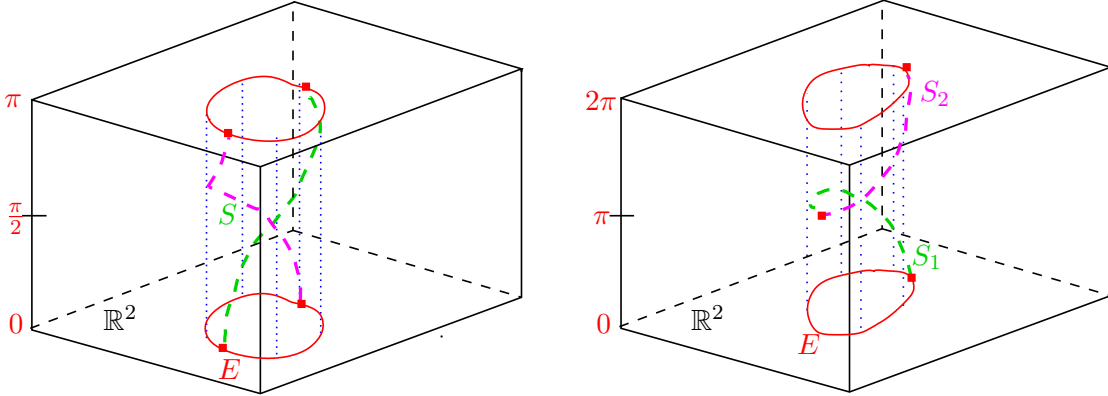


Figure 14: Orienting $G(n, m)$. On the left, we have no orientation. On the right, we have orientation.

Now, let $E \subset \mathbb{R}^n$ be an m -rectifiable set (see Definition 5.2), and at each point x of E , look at $T_x E$, the tangent space at x . We know $T_x E$ exists \mathcal{H}^m -almost

everywhere (a.e.) since E is m -rectifiable, which in turn implies that, except for a measure 0 set, E is contained in the union of countably many Lipschitz functions from \mathbb{R}^m to \mathbb{R}^n .

Definition 8.1. A *rectifiable varifold* is a radon measure μ_E defined on an m -rectifiable set $E \subset \mathbb{R}^n$. Recalling $S \equiv \{(x, T_x E) \mid x \in E\}$, let $A \subset \mathbb{R}^n \times G(n, m)$ and define

$$\mu_E(A) = \mathcal{H}^m(\pi(A \cap S)).$$

Parameterizing $G(n, m)$ as we did in Section 2.3, we obtain the set $S \subset \mathbb{R}^n \times G(n, m)$, as depicted in Figure 15.

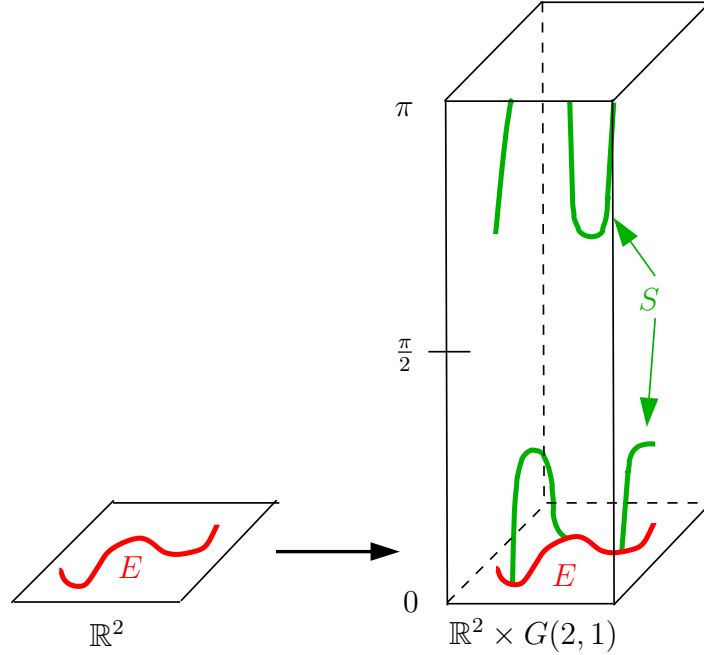


Figure 15: Representation with Varifolds

We can use this representation upstairs in combination with a cubical cover to get a quantized version of the curve that has tangent information as well as position information. For example, if we cover the set $S \subset \mathbb{R}^2 \times G(2, 1)$ with cubes of edge length $\frac{1}{2^d}$ and use this cover as a representation for S , we know position and angle to within $\frac{\sqrt{3}}{2^{d+1}}$. In other words, we can approximate our curve $S \in \mathbb{R}^2 \times G(2, 1)$ by the union of centers of the cubes (with edge length $\frac{1}{2^d}$) intersecting S . This simple idea seems to merit further exploration.

9 Problems and Questions

Problem 9.1. Find a smooth ∂E , with $E \subset \mathbb{R}^n$, such that

$$\mathcal{H}^{n-1}(\partial \mathcal{C}_d^E) / \mathcal{H}^{n-1}(\partial E) = 0 \quad \forall d.$$

Hint: Look at unbounded $E \subset \mathbb{R}^2$ such that $\mathcal{L}^2(E^c) < \infty$.

Problem 9.2. Show that if the **reach** of ∂E is positive, then

$$\liminf_{d \rightarrow \infty} \frac{\mathcal{H}^{n-1}(\partial \mathcal{C}_d^E)}{\mathcal{H}^{n-1}(\partial E)} \geq 1$$

Hint: Examine the map $F : \partial E \rightarrow \mathbb{R}^n$ where $F(x) = x + \eta(x)N(x)$, $N(x)$ is the normal to ∂E at x , and $\eta(x)$ is a positive real-valued function chosen so that *locally* $F(\partial E) = \partial \mathcal{C}_d^E$. Use the Binet-Cauchy Formula to find the Jacobian and then apply the area formula. To do this calculation, notice that at any point $x_0 \in \partial E$ we can choose coordinates so that $T_{x_0}\partial E$ is horizontal (i.e. $N(x_0) = e_n$). Calculate using $F : T_{x_0}\partial E = \mathbb{R}^{n-1} \rightarrow \mathbb{R}^n$ where $F(x) = x + \eta(x)N(x)$.

Problem 9.3. Suppose E has dimension $n - 1$, has positive reach, and is locally regular (in \mathbb{R}^n).

- Find bounds for $\mathcal{H}^n(\mathcal{C}_d^E) / \frac{1}{2^d}$.
- How does this ratio relate to $\mathcal{H}^{n-1}(E)$?

Hint: Use the ideas in Section 6 to calculate a bound on the volume of the tube with thickness $2\frac{\sqrt{n}}{2^d}$ centered on E .

Question 9.1. Can we use the “upstairs” version of cubical covers to find better representations for sets and their boundaries? (Of course, “better” depends on your objective!)

For the following question, we need the notion of the *multiscale flat norm* [22]. The basic idea of this distance, which works in spaces of oriented curves and surfaces of any dimension, is that we can decompose the curve or surface T into $(T - \partial S) + \partial S$, *but* we measure the cost of the decomposition by adding the volumes of $T - \partial S$ and S (not ∂S !). By volume, we mean the m -dimensional volume, or m -volume of an m -dimensional object, so if T is m -dimensional, we would add the m -volume of $T - \partial S$ and the $(m+1)$ -volume of S (scaled by the parameter λ). We get that

$$\mathbb{F}_\lambda(T) = \min_S M_m(T - \partial S) + \lambda M_{m+1}(S).$$

It turns out that $T - \partial S$ is the best approximation to T that has curvature bounded by λ . We exploit this in the following ideas and questions.

Question 9.2. Choose $k \in \{1, 2, 3\}$. In what follows we focus on sets Γ which are one-dimensional, the interior of a cube C will be denoted C^o , and we will work at some scale d , i.e. the edge length of the cube will be $\frac{1}{2^d}$.

Consider the piece of Γ in C^o , $\Gamma \cap C^o$. Inside the cube C with edge length $\frac{1}{2^d}$, we will use the flat norm to

1. find an approximation of $\Gamma \cap C^o$ with curvature by bounded $\lambda = 2^{d+k}$ and
2. the distance of that approximation from $\Gamma \cap C^o$.

This decomposition then is obtained by minimizing

$$M_1((\Gamma \cap C^o) - \partial S) + 2^{d+k} M_2(S) = \mathcal{H}^1((\Gamma \cap C^o) - \partial S) + 2^{d+k} \mathcal{L}^2(S).$$

The minimal S will be denoted S_d . (See Figure 16.)

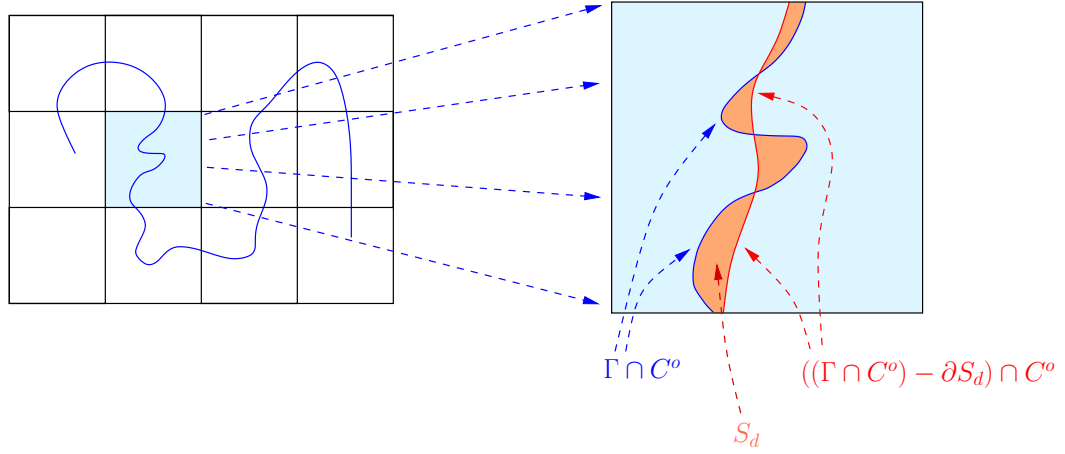


Figure 16: Multiscale flat norm decomposition inspiring the definition of $\beta_{\Gamma}^{\mathbb{F}^d}$

Suppose that we define $\beta_{\Gamma}^{\mathbb{F}}(C)$ by

$$\beta_{\Gamma}^{\mathbb{F}}(C)l(C) = 2^{d+k} \mathcal{L}^2(S_d)$$

so that

$$\beta_{\Gamma}^{\mathbb{F}}(C) = 2^{2d+k} \mathcal{L}^2(S_d).$$

What can we say about the properties (e.g. rectifiability) of Γ given the finiteness $\sum_C (\beta_{\Gamma}^{\mathbb{F}}(3C))^2 l(C)$?

Question 9.3. Can we get an advantage by using the flat norm decomposition as a preconditioner before we find cubical cover approximations? For example, define

$$\mathcal{F}_d^{\Gamma} \equiv \mathcal{C}_d^{\Gamma_d}, \quad \Gamma_d \equiv \Gamma - \partial S_d,$$

$$\text{where } S_d = \operatorname{argmin}_S \left(\mathcal{H}^1(\Gamma - \partial S) + 2^{d+k} \mathcal{L}^2(S) \right).$$

Since the flat norm minimizers have bounded mean curvature, is this enough to force the cubical covers to give us better quantitative information on Γ ? How about in the case in which $\Gamma = \partial E$, $E \subset \mathbb{R}^2$?

10 Further Exploration

There are a number of places to begin in exploring these ideas further. Some of these works require significant dedication to master, and it is always recommended that you have someone who has mastered a path into pieces of these areas that you can ask questions of when you first wade in. Nonetheless, if you remember that the language can always be translated into pictures, and you make the effort to do that, headway towards mastery can always be made. Here is an annotated list with comments:

Primary Varifold References Almgren's little book [2] and Allard's founding contribution [1] are the primary sources for varifolds. Leon Simon's book on Geometric Measure Theory [26] (available for free online) has a couple of excellent chapters, one of which is an exposition of Allard's paper.

Recent Varifold Work Both Buet and Collaborators [5, 6, 4, 7, 8] and Charon and Collaborators [9, 10, 11] have been digging into varifolds with an eye to applications. While these papers are a good start, there is still a great deal of opportunity for the use and further development of varifolds.

Geometric Measure Theory I The area underlying the ideas here are those from geometric measure theory. The fundamental treatise in the subject is still Federer's 1969 *Geometric Measure Theory* [15] even though most people start by reading Morgan's beautiful introduction to the subject [21]. Also recommended are Leon Simon's lecture notes [26], Francesco Maggi's book that updates the classic Italian approach [19] and Krantz and Parks' *Geometric Integration Theory* [17].

Geometric Measure Theory II The book by Mattila [20] approaches the subject from the harmonic analysis flavored thread of geometric measure theory. Some use this as a first course in geometric measure theory, albeit one that does not touch on minimal surfaces, which is the focus of the other texts above. De Lellis' exposition *Rectifiable sets, densities and tangent measures* [14] or Priess' 1987 paper *Geometry of measures in \mathbb{R}^n : distribution, rectifiability, and densities* [24] is also very highly recommended.

Jones’ β In addition to the papers cited in the text, [16, 23, 18, 25] there are related works by David and Semmes that would be recommended. See for example [13]. There is also the applied work by Gilad Lerman and his Collaborators that is often inspired by Jones’ β and his own development of Jones’ ideas in [18]. See for example [12, 28, 27, 3].

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